# On the Parameterized Complexity of Learning Monadic Second-Order Formulas

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# Understanding the Problem

# Related Problems

**Model Checking**

**Input**: Graph $\mathcal{G}$, $\varphi$

**Problem**: $\mathcal{G} \models \varphi$?

**Enumeration**

**Input**: Graph $\mathcal{G}$, $\varphi(x_1, x_2)$

**Problem**: Enumerate all $(v_1, v_2)$ such that $\mathcal{G} \models \varphi(v_1, v_2)$

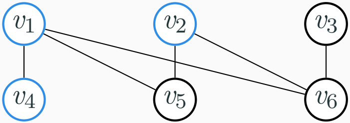
**Consistent Learning**

**Input**: Graph $\mathcal{G}$, $\mathbb{S} = \{((v_1, v_2), +), ((v_3, v_4), -)\}$

**Problem**: Find $\varphi$ such that $\mathcal{G} \models \varphi(v_1, v_2)$ and $\mathcal{G} \not\models \varphi(v_3, v_4)$

# Example

$\mathcal{A}:$




$$\mathbb{S} = \{((v_1, v_2), +), ((v_2, v_3), +), ((v_3, v_5), -)\}$$

Find $\varphi(x_1, x_2)$

$$\varphi(x_1, x_2) = ((x_1 = v_1) \wedge (x_2 = v_2)) \vee ((x_1 = v_2) \wedge (x_2 = v_3))?$$

Limit the number of parameter variables!

Find $\varphi(x_1, x_2, y_1)$ and an assignment for $y_1$

# Problem Definition MSO-Consistent-Learn

Input:

- $\tau$-structure $\mathcal{A}$ of universe $A$
- training set $\mathbb{S} \subseteq A^d \times \{+,-\}$
- $d, \ell, q \in \mathbb{N}$

Output: $h_{\varphi,\bar{v}}$

- $\varphi(\bar{x}, \bar{y}) \in \text{MSO}[\tau, q, d+\ell, 0]$

  instance variables ($\bar{x}$)     parameter variables ($\bar{y}$)

- parameter setting $\bar{v} \in A^\ell$
- consistent

  $(\bar{w}_1, +) \in \mathbb{S}$ then $\mathcal{A} \models \varphi(\bar{w}_1, \bar{v})$

  $(\bar{w}_2, -) \in \mathbb{S}$ then $\mathcal{A} \not\models \varphi(\bar{w}_2, \bar{v})$

# Parameterized Complexity

Parameterized complexity of a problem $Q$ with input $x$ and parameterization $\kappa(x)$:

Input $x$

$\kappa(x)$

Is there an algorithm for $Q$ with runtime $f(\kappa(x)) \cdot p(|x|)$?
Then $(Q, \kappa)$ is fixed-parameter tractable.

For MSO-Consistent-Learn we choose

$$\kappa = |\tau| + d + \ell + q$$

$$+\text{tw}(\mathcal{A})?$$

# Tractability

# Tractability - Overview

Previous result:

- MSO-Consistent-Learn for $d = 1$ on strings (Grohe, Löding, Ritzert, 2017)
- MSO-Consistent-Learn for $d = 1$ on trees (Grienenberger, Ritzert, 2019)

New results:

- MSO-Consistent-Learn for $d = 1$ on relational structures of bounded tree-width
- MSO-Consistent-Learn for $d = 1$ on graphs of bounded clique-width
- MSO-PAC-Learn for higher dimensions $d \geq 1$ on relational structures of bounded tree-width

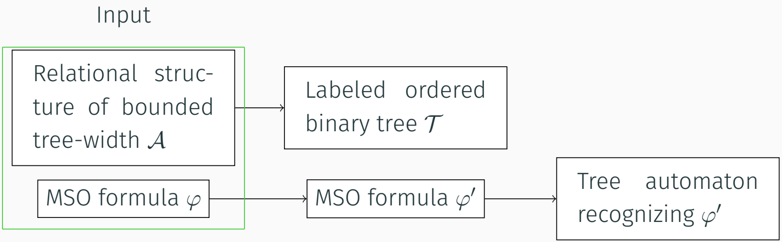
# Tractability - How?

Find formula $\varphi(\bar{x}, \bar{y})$ and parameter assignment $\bar{v} \in A^\ell$

The size of MSO$[\tau, q, d + \ell, 0]$ is small

$\Rightarrow$ Finding $\varphi$ is easy.

Main Problem: Finding the parameter assignment $\bar{v} \in A^\ell$




$$\mathcal{A} \models \varphi \quad \leftrightarrow \quad \mathcal{T} \models \varphi'$$

# Hardness

# Hardness

How hard is learning compared to model checking?

Previous Results:

- FO-Learn is as hard as FO-MC
  (van Bergerem et al., 2022)

MSO-Mc (Model Checking)

| | |
|---|---|
| **Input**: | Graph $\mathcal{G}$, monadic second-order formula $\varphi$ |
| **Problem**: | Does $\mathcal{G} \models \varphi$ hold? |

# $q$-Type

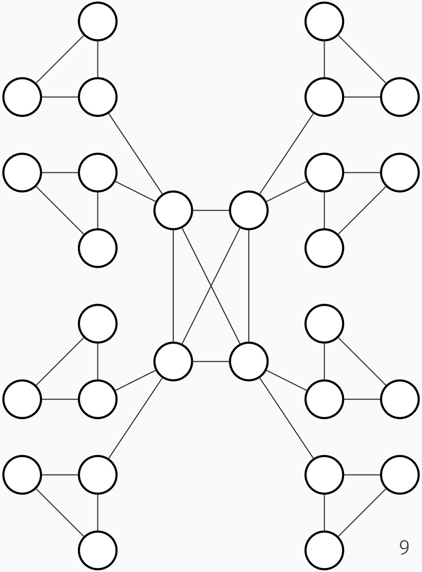
Example: $\exists x\ \psi(x)$

$tp_{q,\tau}(\mathcal{A}, a) = \{\varphi(x) \mid \mathcal{A} \models \varphi(a)\}$
for $\varphi \in$ MSO$[\tau, q, 1, 0]$

Set of representatives:
$R = \{v_1, v_3, v_4\}$

Run Consistent-Learn
oracle on $\mathcal{G}$
with $\mathbb{S} = \{(v_1, +), (v_2, -)\}$

What about $\exists X\ \psi(X)$?



# Results

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fixed parameter tractable

PAC learner

MSO-PAC-LEARN on

- Relational structures of bounded tree-width (including trees)
- Graphs of bounded clique-width (including strings)

consistent learner

1D-MSO-LEARN

- Strings [$*^1$]
- Trees [$*^2$]
- Relational structures of bounded tree-width
- Graphs of bounded clique-width

Hardness:

- **Work in progress** (But probably as hard as MSO-MC)